Pál Erdös and Zygmunt Zahorski

Edyta Hetmaniok, Damian Słota and Roman Wituła

Probably some Readers ask themselves what connected these two great mathematicians (who were of the same age, coincidentally). Taking into considerations the interests of P. Erdös (see for example [1, 10], and everyone is quite well informed that, above all, it is about the discrete mathematics and the number theory) our suspicions fall on the trigonometric series and that is pretty right.

Professor Zahorski formulated in 1957 in Colloquium Mathematicum (see [12]) the problem of determination of the best possible estimation from above of the following integral

$$\int_0^{2\pi} |\cos k_1 x + \cos k_2 x + \ldots + \cos k_n x|\, dx,$$

where $0 < k_1 < k_2 < \ldots < k_n$ are integers. He observed that by applying the Schwartz inequality and by using the orthogonality of trigonometric system we get the estimation $c\sqrt{n}$. Moreover, Zahorski conjectured that $c \log k_n$ is valid as well.

P. Erdös solved Zahorski's problem and the conjecture. In paper [4] published in 1960 in Colloquium Mathematicum he obtained the following two results.

At first, Erdös proved that for every $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that

$$\int_0^{2\pi} \left| \sum_{k=1}^{n} \cos k^2 x \right| dx > c\, n^{\frac{1}{2}-\varepsilon},$$

for every $n \in \mathbb{N}$.

Secondly, Erdös proved the existence of increasing sequence $\{k_i\}_{i=1}^{\infty}$ of natural numbers such that

$$\int_0^{2\pi} \left| \sum_{i=1}^{n} \cos k_i x \right| dx = \sqrt{\pi}\sqrt{k_n} + o(\sqrt{k_n})$$

E. Hetmaniok, D. Słota, R. Wituła
Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland,
e-mail: {edyta.hetmaniok, damian.slota, roman.witula}@polsl.pl

R. Wituła, D. Słota, W. Hołubowski (eds.), *Monograph on the Occasion of 100th Birthday Anniversary of Zygmunt Zahorski.* Wydawnictwo Politechniki Śląskiej, Gliwice 2015, pp. 63–66.

which proves that $O(\sqrt{k_n})$ is the best estimation (so Zahorski's conjecture was not true). We note that from Zahorski's observation it follows that

$$\limsup \frac{k_n}{n} < \infty.$$

Four redactional remarks

1. Let $n \in \mathbb{N}$, $a_1, a_2, \ldots, a_{n-1} \in \mathbb{R}$ if $n \geqslant 2$ and at least one $a_k \neq 0$. Then the function

$$f(x) = \cos nx + a_{n-1}\cos(n-1)x + \ldots + a_2 \cos 2x + a_1 \cos x$$

takes the positive as well as the negative values (the elementary proof is given in [9], see Theorem 1.7 and Remark 1.8). This fact follows also easily from the classical Sturm-Hurwitz Theorem on the lower bound of the number of roots of trigonometric polynomial (see [5, 7, 11]).

Theorem (C. Sturm, 1836, A. Hurwitz, 1903). *Let*

$$f(x) = \sum_{k=n}^{N} (a_k \cos kx + b_k \sin kx),$$

where $n, N \in \mathbb{N}$ and a_k, b_k are real numbers. Then the number of sign changes of f is at least equal to $2n$.

2. Let $n \in \mathbb{N}$. Let us put

$$c_n := \min\left\{ \sum_{k=1}^{n} |\cos kx| \,:\, x \in \mathbb{R} \right\}.$$

Then $c_n = \lfloor n/2 \rfloor$ for every $n > 2$ with exceptions

$$c_4 = \sum_{k=1}^{4} \left|\cos k\frac{\pi}{6}\right| = 1 + \frac{\sqrt{3}}{2} < 2$$

and

$$c_6 = \sum_{k=1}^{6} \left|\cos k\frac{\pi}{10}\right| \approx 2.97 < 3.$$

It was proven by M.B. Munoz, E.F. Moral, J.B. Sagasta, L. Mercedes, M. Mercedes, S. Benito, J.B.L. Bunuel in Crux Mathematicorum (see [9], Theorems 10.3–10.9).

3. Paul Cohen proved in [2] the following theorem.

   Theorem. *For some $k > 0$ and for all $N \in \mathbb{N}$, $N \geq 3$, we have*

$$\int_0^{2\pi} \left| \sum_{j=1}^{N} c_j e^{i\, n_j x} \right| dx > k \left(\frac{\log N}{\log \log N} \right)^{1/8}, \tag{1}$$

   whenever n_j are the distinct integers (also negative) and $c_j \in \mathbb{C}$ satisfy condition $|c_j| \geq 1$.

   Corollary 1. *For every increasing sequence of nonnegative integers $n_1 < n_2 < \ldots < n_N$ we have*

$$\int_0^{2\pi} \left| \sum_{j=1}^{N} \cos n_j x \right| dx > k \left(\frac{\log N}{\log \log N} \right)^{1/8}, \tag{2}$$

   for some $k > 0$ and $N \in \mathbb{N}$, $N \geq 2$.

   Corollary 2. *For every increasing sequence $\{n_j\}_{j=1}^{\infty}$ of nonnegative integers we obtain*

$$\lim_{N \to \infty} \int_0^{2\pi} \left| \sum_{j=1}^{N} \cos n_j x \right| dx = \infty.$$

   Harold Davenport in [3] proved that power $1/8$ in the right hand side of (1) (and, in consequence, in (2) as well) could be replaced by $1/4$ and the constant k by number $1/8$.

4. In 2014 year Ferenc Móricz published in Notices of the AMS a very interesting and beautiful paper [8] dedicated to the memory of Pál Erdös on the 100th anniversary of his birthday. Subject matter, discussed in this paper, especially the generalized Rademacher-Menshov maximal moment inequality is directly connected with the object of our note. Extremely intriguing is also the description by Móricz of some Erdös theorem on the convergence a.e. of trigonometric series satisfying the so-called (B_2) Erdös condition, which is the generalization of the Kolmogorov Theorem [6] on the convergence a.e. of the lacunary trigonometric series. It suggests another connotation to the Kolmogorov – Zahorski mathematical relations (especially in reference to other Kolmogorov Theorem proved by Zahorski in paper [13]).

Bibliography

1. Bollobas B.: *To prove and conjecture: Paul Erdös and his mathematics.* Amer. Math. Monthly **105**, no. 3 (1998), 209–237.
2. Cohen P.J.: *On a conjecture of Littlewood and idempotent measures.* Amer. J. Math. **82** (1960), 191–212.

3. Davenport H.: *On a theorem of P.J. Cohen.* Mathematica **7** (1960), 93–97.
4. Erdös P.: *About an estimation problem of Zahorski.* Colloq. Math. **7**, fasc. 2 (1960), 167–170.
5. Katriel G.: *From Rolle's Theorem to the Sturm-Hurwitz Theorem.* ArXiv:math/0308159.
6. Kolmogorov A.N.: *Une contribution à l'étude de la convergence des séries de Fourier.* Fund. Math. **5** (1924), 96–97.
7. Martinez-Maure Y.: *Les multiherissons et le theoreme de Sturm-Hurwitz.* Arch. Math. **80** (2003), 79–86.
8. Móricz F.: *Moment inequalities for maxima of partial sums in probability with applications in the theory of orthogonal series.* Notices Amer. Math. Soc. **61**, no. 6 (2014), 576–585.
9. Palarz M.: *Trigonometric polynomials.* Graduation dissertation supervised by R. Wituła, Pol. Śl., Gliwice 2003 (in Polish).
10. Sós V.T.: *Paul Erdös, 1913–1996.* Aequationes Math. **54** (1997), 205–220.
11. Tabachnikov S.: *Proofs (not) from the Book.* Math. Intelligencer **36**, no. 2 (2014), 9–14.
12. Zahorski Z.: *Problem 168.* Colloq. Math. **4** (1951), 241.
13. Zahorski Z.: *Une série de Fourier permutée d'une fonction de classe L^2 divergente presque partout.* C. R. Acad. Sci. Paris **251** (1960), 501–503.

Jan Lipiński and Zygmunt Zahorski